Synopsis: Review of the course MATH-213: Differential Geoemtry I

immediate

Below you can find a list of the material that was covered in class, which you would have to review for the exam:

1. *Vector geometry in Euclidean Spaces*

Geometry of Euclidean vector spaces: inner product, orthogonal transformations, isometries, the notion of orientation of bases, as well as the various products of vector geometry.

2. *Curves in Euclidean Space and the Plane*

You should know the basic definitions concerning parametrized curves, in particular: velocity, speed, regular curves, biregular curves, osculating plane, lengths, arc-length parameter, reparametrization of a curve, natural parametrization, closed curve, simple curves etc.

- Understand what a geometric notion is (i.e. independent of parametrization, as opposed to a kinematic one) and know how to prove that a given notion is geometric (for example, the length or the curvature of a curve).
- Understand the notions related to the curvature of a curve: the curvature vector, the acceleration formula, the osculating circle. Understand and know how to use the Serret–Frenet formulas.
- Be able to compute, in concrete examples, the Frenet frame, curvature, and torsion of a given curve. Understand the geometric meaning of these notions.
- Understand the statement and the proof of the fundamental theorem of the theory of curves in space, as well as the version of this theorem in the plane.

You should understand the statement of the four vertex theorem for simple closed curves in the plane, but you don't have to spend time on the proof.

3. *Differential Calculus and Submanifolds.*

One should be able to deal comfortably with notions such as C^k maps, differentiable maps in the sense of Fréchet, the differential of a map, the Jacobian, the gradient, the notion of the rank of a differentiable map, and the notions of immersion and submersion, etc.

- Understand the notion of a system of curvilinear coordinates on a domain of $\mathbb{R}^n$

- Know the statements of the inverse function theorem, implicit function theorem and the constant rank theorem, and know how to use these theorems (but there is no need to know the proofs for this exam).

It is important to understand the definition of what a submanifold of $\mathbb{R}^n$ is, and see in certain examples how one can prove that a certain set is a submanifold (often using the constant rank theorem). Another key concept to understand thoroughly is that of the tangent space at a point of a submanifold, followed by that of a differentiable map between two submanifolds and the definition of the differential of a (sufficiently regular) map between submanifolds. Also, one needs to be comfortable with dealing with local parametrizations of submanifolds.

4. *Geometry of Submanifolds.*

Understand the notions of intrinsic and extrinsic distances on a submanifold, how intrinisic isometries are characterized by maps whose differentials are linear isometries between the corresponding tangent spaces. Understand the notion of the metric tensor associated to a given local parametrization of a submanifold. Study carefully the examples that we saw in the class and exercise sheets dealing with the computation of the metric tensor associated to given submanifolds (for example, that of surfaces of revolution).

You should also be able, using the expression for the metric tensor, to compute lengths of curves, angles between tangent vectors and areas/volumes of domains in submanifolds, as well as the notion of integration on a submanifold.

You should also understand the formulas for how the metric tensor changes under a change of parametrization, as well as the characterization for an isometry between two submanifolds in terms of the expression of the corresponding metric tensors in local parametrizations.

5. *The geometry of surfaces.* The surfaces in $\mathbb{R}^3$ constitute in this course the prime example of submanifolds for which we studied their extrinsic geometry in a lot of detail. For the exam, you should:

- Understand the notion of a (co-)oriented surface and the Gauss map.
- Know the notion of a geodesic of a surface and its properties. in particular, you should understand the statement that a curve that is a minimizer of the length among curves connecting two given points on a surface is a geodesic.
- Know the definition of the Darboux frame of a curve drawn on a surface, the notions of geodesic curvature, normal curvature, and torsion, as well as the Darboux equations. The geometric interpretation of these notions was covered in the exercises.
- Thoroughly understand Meusnier's Theorem and its proof.
- The curvature of a surface is described using the shape operator (Weingarten map) and the second fundamental form. It is important to master these concepts, their fundamental properties, and the links between them.
- Understand the notions of parabolic, hyperbolic, elliptic, and umbilic points of a surface.
- Be able to compute the Gaussian and mean curvatures of a parametrized surface in concrete examples.

– Understand the statement and proof of the theorem stating that any connected surface with all its points being umbilic is either part of a sphere or part of a plane.

– Understand the definition of the Christoffel symbols associated to a parametrization of a surface, as well as the proof of the Levi-Civita formula for their expression.

– You should understand Gauss's Theorema Egregium, as well as the statement and the proof of the rigidity results that we saw for surfaces (i.e. that the metric tensor and second fundamental form uniquely determine a surface up to a rigid motion).

Final Remarks.

Being able to perform calculations with the objects of differential geometry is part of the objectives of the course. In the context of the exam, you will therefore have to carry out calculations. If you do them carefully, the resulting expressions simplify quite significantly, and the answers to the questions should not be monstrously complicated formulas. You should be also able to repeat some of the simple proofs we saw in class; you will not be asked to repeat very long proofs (such as that of the four vertex theorem, which has been omitted from the exam material), but the techniques we saw for the various proofs in class could appear as parts of exercises.

In the exam, you will be allowed to carry with you two pages of cheat-sheets (on paper, they could be printed from a tablet, but you will not be allowed to have a screen with you).